Lecture 6 02/05/2018

Review of Electrostatics (Cont'd)

Spherical Coordinates

$$\nabla^2 \Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Phi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

$$\Phi(r,\theta,\phi) = R(r)\,P(\theta)\,Q(\phi) \Rightarrow \frac{1}{R(r)}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{r^2\sin\theta}\frac{1}{P(\theta)}$$

$$\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{1}{Q}\frac{d^2Q}{d\phi^2} = 0 \Rightarrow \underbrace{\frac{1}{R(r)\,r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)}_{\text{function of } r \text{ only}} +$$

$$\frac{1}{r^2}\underbrace{\left[\frac{1}{\sin\theta P}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \frac{1}{\sin^2\theta\, Q}\frac{d^2Q}{d\phi^2}\right]}_{\text{function of } \theta,\phi \text{ only}} = 0$$

Each of these functions must be a constant:

$$\begin{cases} \dfrac{1}{Q}\dfrac{d^2Q}{d\phi^2} = -m^2 \Rightarrow Q(\phi) \propto e^{\pm im\phi} \\ \dfrac{1}{\sin\theta P}\dfrac{d}{d\theta}\left(\sin\theta\dfrac{dP}{d\theta}\right) - \dfrac{m^2}{\sin^2\theta} + \ell(\ell+1) = 0 \\ \dfrac{1}{r^2 R}\dfrac{d}{dr}\left(r^2\dfrac{dR}{dr}\right) - \dfrac{\ell(\ell+1)}{r^2} = 0 \Rightarrow R \propto r^{\ell},\ r^{-(\ell+1)} \end{cases}$$

Solutions to the second equation are associated Legendre functions.

$P_\ell^m(\cos\theta)$, $Q_\ell^m(\cos\theta)$. However, considering the entire range of the polar angle $0 \le \theta \le \pi$, these functions are singular at $\theta = 0, \pi$ unless $\ell = 0, 1, 2, \ldots$ and $m = -\ell, -\ell+1, \ldots, \ell-1, \ell$. Even then only $P_\ell^m(\cos\theta)$ is finite, which is called the associated Legendre polynomial:

$$P_\ell^m(\cos\theta) = (-1)^m \sin^{|m|}\theta \frac{d^{|m|}}{d\cos\theta^{|m|}} P_\ell(\cos\theta)$$

$P_\ell(x)$ are the Legendre polynomials that satisfy the following differential equation:

$$(1-x^2)\frac{d^2 P_\ell}{dx^2} - 2x\frac{dP_\ell}{dx} + \ell(\ell+1)P_\ell = 0$$

They are given by the Rodriguez formula:

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell}(x^2-1)^\ell$$

The orthogonality relation for P_ℓ is:

$$\int_{-1}^{+1} P_\ell(x) P_{\ell'}(x)\, dx = \frac{2\delta_{\ell\ell'}}{2\ell+1}$$

We can write $P_\ell^m(\cos\theta)\, e^{im\phi}$ in terms of the "spherical harmonics":

$$Y_{\ell m}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\theta)\, e^{im\phi}$$

Some of the useful properties of $Y_{\ell m}$ are as follows:

$$Y_{\ell m}(\theta,\phi) = (-1)^m Y^*_{\ell m}(\theta,\phi)$$

$$\iint Y^*_{\ell m}(\theta,\phi)\, Y_{\ell' m'}(\theta,\phi)\, \sin\theta\, d\theta\, d\phi = \delta_{\ell\ell'}\,\delta_{mm'} \qquad \text{(orthogonality)}$$

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell} Y^*_{\ell m}(\theta,\phi)\, Y_{\ell m}(\theta',\phi') = \frac{1}{\sin\theta}\,\delta(\theta-\theta')\,\delta(\phi-\phi')$$

We can also write $\frac{1}{|\vec{x}-\vec{x}'|}$ in terms of the spherical harmonics:

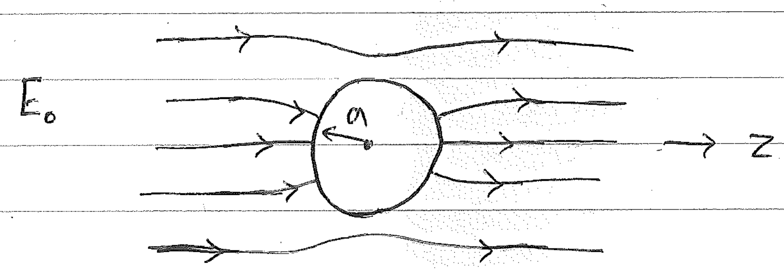
$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{\ell=0}^{\infty} \frac{r_<^{\ell}}{r_>^{\ell+1}} \cdot \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{+\ell} Y^*_{\ell m}(\theta,\phi)\, Y_{\ell m}(\theta',\phi')$$

Here $r_<$ is the smaller of $|\vec{x}|, |\vec{x}'|$, while $r_>$ is the larger of the two.

The general solution for Φ can then be written as:

$$\Phi(r,\theta,\phi) = \sum_{\ell,m}\left[A_{\ell m} r^{\ell} + B_{\ell m} r^{-(\ell+1)}\right] Y_{\ell m}(\theta,\phi)$$

Example: Conducting sphere in a uniform electric field.



Choosing the z axis as shown above, we can exploit the azimuthal symmetry, which implies no ϕ dependence. This simplifies the as only terms with $m=0$ are now present in Φ:

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-(l+1)}\right) Y_{lm}(\theta,\phi)$$

Asymptotic behavior of the electric field implies that $A_l = 0$ for $l \geq 2$. Also, since the sphere is isolated and has no net charge, we have $B_0 = 0$. Having an equipotential surface at $r=a$ requires that:

$$A_1 a + B_1 a^{-2} = 0 \Rightarrow B_1 = A_1 a^3$$

Note that $\Phi \to -E_0 z$ as $|z| \to \infty$. Considering that $z = r\cos\theta$, we then find:

$$\boxed{\Phi(r,\theta) = -E_0 r\cos\theta + E_0 \frac{a^3}{r^2}\cos\theta} \quad *$$

This is the superposition of the potential due to a uniform electric field and that due to an electric dipole. This can be seen by calculating the induced surface charge density on the sphere:

$$\sigma(\theta) = \epsilon_0 E_n(\theta) = -\epsilon_0 \frac{\partial \Phi}{\partial r}\Big|_{r=a} = 3\epsilon_0 E_0 \cos\theta$$

The dipole moment of the sphere thus follows:

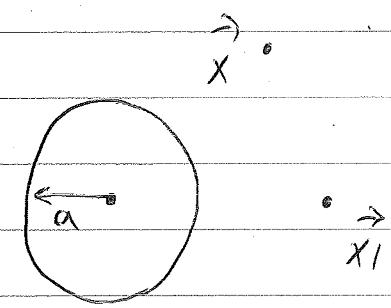
$$\vec{P} = a^3 \int_0^{2\pi}\int_0^{\pi} \sigma(\theta) \cos\theta \sin\theta \, d\theta \, d\phi \, \hat{z} = 4\pi \varepsilon_0 E_0 a^3 \hat{z}$$

The electric potential due to such a dipole is $\frac{\vec{P}\cdot\vec{x}}{r^2}$, which gives exactly the same result as in the second term on the right-hand side of * in above.

We now consider applications of the spherical harmonics in computing Green's functions for problems with spherical boundaries.

Green's Functions for the Exterior Problem

As we saw before:



$$G(\vec{x}, \vec{x1}) = \frac{1}{|\vec{x} - \vec{x1}|} + \tilde{G}(\vec{x}, \vec{x1})$$

Where:

$$\nabla^2 \tilde{G}(\vec{x}, \vec{x1}) = 0$$

The general solution for $\tilde{G}$ is given by:

$$\tilde{G}(\vec{x}, \vec{x1}) = \sum_{l,m} B_l \, r^{-(l+1)} \, Y_{lm}(\theta, \phi)$$

We note that r^{ℓ} terms vanish due to local nature of the boundary as $\bar{G}$ is the potential due to the induced charges on the boundary.

Hence:

$$G(\vec{x},\vec{x}') = \sum_{\ell,m} B_\ell r^{-(\ell+1)} Y_{\ell m}(\theta,\phi) + \frac{1}{|\vec{x}-\vec{x}'|} = \sum_{\ell,m} B_\ell r^{-(\ell+1)} Y_{\ell m}(\theta,\phi)$$

$$+ \sum_{\ell,m} \frac{r_<^{\ell}}{r_>^{\ell+1}} \cdot \frac{4\pi}{2\ell+1} Y_{\ell m}(\theta,\phi) Y^*_{\ell m}(\theta',\phi') = \sum_{\ell,m} \left[\frac{4\pi}{2\ell+1} \frac{r_<^{\ell}}{r_>^{\ell+1}} Y^*_{\ell m}(\theta',\phi') \right.$$

$$\left. + B_\ell \frac{1}{r^{\ell+1}} \right] Y_{\ell m}(\theta,\phi)$$

For the <u>Dirichlet problem</u>, we must have $G_D(\vec{x},\vec{x}')\Big|_{r=a} = 0$.

This requires that:

$$\frac{4\pi}{2\ell+1} \frac{a^{\ell}}{r'^{\ell+1}} Y^*_{\ell m}(\theta',\phi') + B_\ell \frac{1}{a^{\ell+1}} = 0 \quad (r_< = a,\ r_> = r')$$

Therefore:

$$\boxed{B_\ell = \frac{-4\pi}{2\ell+1} \frac{a^{2\ell+1}}{r'^{\ell+1}} Y^*_{\ell m}(\theta',\phi')}$$

For the <u>Neumann problem</u>, we must have $\frac{\partial G_N(\vec{x},\vec{x}')}{\partial r}\Big|_{r=a} = 0$.

This condition determines the coefficients B_ℓ in this case.

Green's Functions for the Interior Problem

In this case, the terms $A_\ell r^\ell$ are present in the expression for $\bar{G}(\vec{x},\vec{x}')$. The Dirichlet problem is similar to that for the exterior problem as the condition $G_D(\vec{x},\vec{x}')\big|_{r=a}=0$ determines the coefficients A_ℓ. However, the Neumann problem is more involved in this case. The reason being that $\frac{\partial G_N(\vec{x},\vec{x}')}{\partial r}\Big|_{r=a} = \frac{-4\pi}{4\pi a^2} = -\frac{1}{a^2}$. This affects the $\ell=0$ and $m=0$ contribution, which is a constant. For $\ell \neq 0$, the coefficients A_ℓ are determined analogously to the exterior problem.

Finally, let us discuss the eigenfunction approach for finding the Green's functions. Consider the eigenfunctions of the Laplace operator with the appropriate boundary conditions (Dirichlet or Neumann) in the volume of interest:

$$\nabla^2 f_\lambda(\vec{x}) = \lambda f_\lambda(\vec{x})$$

This is basically a Sturm-Liouville problem for which a complete orthonormal set of eigenfunctions is possible to find. In d-dimensions, the eigenfunctions and their corresponding eigenvalues may be labelled by "d" indices (integers if the volume is bounded, continuous otherwise).

Thus, in one-dimension, we can write:

$$\frac{d f_n(\eta)}{d\eta^2} = \lambda_n f_n(\eta)$$

Then it can be shown that:

$$G(\vec{x}, \vec{x}') = -4\pi \sum_n \frac{f_n(\vec{x})\, f_n^*(\vec{x}')}{\lambda_n}$$

We note that the sum is replaced by an integral if λ varies continuosly. Generalization to two- and three-dimensions is straightforward:

$$G(\vec{x}, \vec{x}') = -4\pi \sum_m \sum_n \frac{f_{mn}(\vec{x})\, f_{mn}^*(\vec{x}')}{\lambda_{mn}} \qquad \text{(2-dimension)}$$

$$G(\vec{x}, \vec{x}') = -4\pi \sum_l \sum_m \sum_n \frac{f_{lmn}(\vec{x})\, f_{lmn}^*(\vec{x}')}{\lambda_{lmn}} \qquad \text{(3-dimension)}$$